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ELEMENTARY ALLOCATION MODELS...
how to use them in forestry operations

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ABSTRACT: How best to allocate available resources is a common problem in forestry operations. The optimum or most efficient solution to the problem is best achieved by using complex mathematical techniques and computers. But now that simplified versions of these techniques have been developed, it is possible to use some allocation models without computing equipment. The use of two allocation models–transportation and assignment–is explained step-by-step and illustrated with examples of their application to forestry problems.
RETRIEVAL TERMS: resource allocation models; Vogel's approximation method; allocation models; assignment models.
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Problems of allocation arise whenever you have several activities to perform, but cannot do each in the most effective way. You may, for example, need to allocate resources. In some situations, supply equals demand; but in others, supply is less than demand. Allocation problems are most easily solved by computers. But it may be years before it is as easy to obtain a desk computer as it is to get an adding machine. Even without computing equipment, you can solve many elementary allocation problems. And all you need is paper and pencil. This is made possible by simplified versions of complex mathematical techniques that have been developed recently.

To illustrate the use of three allocation techniques or models, I have taken examples of their application to forestry operations. But these problem-solving models can be used in just about any kind of allocation activity. How each model is used is explained step-by-step.

TRANSPORTATION MODEL

The general form of allocation problems has been called the *transportation model* because it is often used in solving the problem of transporting things from one place to another. In the general form of the model, supply equals demand.[1]

From each origin to each destination, there will usually be a different distance and different per-unit shipping cost associated with each combination of origin and destination. The aim is to ship all items so that total or over-all shipping cost is minimized. The best combination of routes is not always obvious–it may be necessary to ship one item at a moderate cost in order to achieve a low cost for a second item.

A trial-and-error approach to the solution probably would not help—the number of possible ways of allocating resources to activities can be mathematically overwhelming. We could, in theory, list all possible choices. For example, there are 8!=40,320 ways of allocating eight items, one apiece, to eight demands. Supply must equal demand, but the number of supply points does not have to equal the number of demand points.

To illustrate the use of the transportation model, let us consider a problem that might arise during slash burning in the Douglas-fir region. The problem is to transport chemical fire retardant from warehouse to slash block in the most optimal way. The retardants help in controlling slash burns after a fire season. Certain men are kept and organized into crews to prepare slash blocks for burning and to conduct the burning operation over a 16-week period in late fall and early spring. The total burning job is supervised by an assistant fire control officer, who in turn receives administrative supervision and guidance from a fire control staff specialist. We will assume that both men are experienced and skilled in fire control and personnel supervisory techniques. Four of the slash burning crews plan to use chemical fire retardants as an aid in controlling their slash fires. The retardant is available in 50-pound bags. The needs of each crew are:

Crew	*No. of bags*
1	120
2	50
3	50
4	280
Total	500

The number of bags in each warehouse is:

Warehouse	*No. of bags*
1	100
2	60
3	140
4	200
Total	500

In our example the relation between supply and demand can be shown in *matrix* form; that is, the elements making up the relationship are arrayed in rows and columns (fig. 1). Costs can be any unit of measure. In this example, "costs" will be expressed in terms of the round-trip distance between warehouse and crew headquarters—the distance that the bags must be transported. Another approach might be to consider the shipping cost per bag, ton, or some other unit. Then the total cost would be the product of the quantity of items times the shipping cost. The aim is to minimize the number of miles that have to be traveled to empty the warehouses and supply the crews.

A few minutes' work will show that the problem has a great many possible solutions—some of them much better than others. To solve this problem, we turn to Vogel's Approximation Method (VAM)—a relatively simple approximation procedure that will yield a near optimum solution in most allocation problems.[2]

The six steps in the VAM are as follows:

1. *Compute the difference between the smallest and second smallest elements (cost or profit) for each row and each column of the matrix (fig. 1). Place these values around the edge of the matrix.*

In our example, the difference in the first column is 38 – 22 = 16.

2. *Select the row or column with the greatest difference (fig. 2) and assign as much of the resource as possible to the square at the lowest cost.*

Row (warehouse) 1 is the closest (lowest cost) to crew 3 so we allot as much as possible (50 bags) to crew 3.

3. *Cross out the row or column completely used or satisfied by the assignment.*

Crew 3 has been completely satisfied (they have their requirement of 50 bags); however, 50 bags still remain at warehouse 1.

4. *Redetermine the differences as may be necessary, but omit the row(s) or column(s) crossed out (fig. 3).*

The greatest difference (42) is found in row 2 after column 3 has been crossed out. It consists of 74 – 32 = 42.

5. *Repeat steps 2 through 4 until all the assignments are made (figs. 4,5).*
6. *After the fourth assignment has been made in our example, the VAM can no longer be applied because only one column (crew 4) remains unsatisfied. However, the assignment can be easily done by inspection (fig. 5).*

We simply assign all of the remaining bags in the warehouse 1, 2, and 4 to crew 4. Crew 4 is now satisfied, and all warehouses have been emptied.

	(16)	(2)	(52)	(20)		
Warehouse \ Crew	1	2	3	4	No. bags of retardant available	
1	78	18	10	54	100	(8)
2	74	32	62	76	60	(30)
3	38	16	74	8	140	(8)
4	22	28	70	28	200	(6)
No. bags of retardant required	120	50	50	280	Σ500	

Figure 1.—*Standard allocation model. Matrix for solving problem of shipping bags of retardants from warehouses to crew headquarters. Number of bags available is shown at right; number of bags required by projects is shown at bottom. Costs are the number of miles (round trip) between warehouse and project headquarters and are given in the square. Number in parentheses are differences between the smallest and the next smallest elements (costs or profit) in each row and column.*

	(16)	(2)	(52)	(20)		
Warehouse \ Crew	1	2	3	4	No. bags of retardant available	
1	78	18	10 (50)	54	100	(8)
2	74	32	62 ×	76	60	(30)
3	38	16	74 ×	8	140	(8)
4	22	28	70 ×	28	200	(6)
No. bags of retardant required	120	50	50	280	Σ500	

Figure 2.—*After differences within columns and rows are computed, select the row or column with the greatest difference. Assign as many bags of retardants to the square at the lowest cost (fewest miles distance). Cross out the triangles satisfied by the assignment.*

	(16)	(2)	(52)	(20)		
Warehouse \ Crew	1	2	3	4	No. bags of retardant available	
1	78	18 ×	10 50	54	100	(36)
2	74	32 (50)	62 ×	76	60	(42)
3	38	16 ×	74 ×	8	140	(8)
4	22	28 ×	70 ×	28	200	(6)
No. bags of retardant required	120	50	50	280	Σ500	

Figure 3.—*In the fourth step, redetermine the differences as may be necessary, but omit any row or column crossed out in earlier steps.*

	(16)	(2)	(52)	(20)		
Warehouse \ Crew	1	2	3	4	No. bags of retardant available	
1	78	18 ×	10 50	54	100	(24)
2	74	32 50	62 ×	76	60	(2)
3	38 ×	16 ×	74 ×	8 (140)	140	(30)
4	22	28 ×	70 ×	28	200	(6)
No. bags of retardant required	120	50	50	280	Σ500	

Figure 4.—*Repeat second through fourth steps until all assignments and retardants to crew headquarters are made.*

	1	2	3	4	No. bags of retardant available
Warehouse \ Crew					
1	78 ×	18 ×	10 50	54 (50)	100
2	74 ×	32 50	62 ×	76 (10)	60
3	38 ×	16 ×	74 ×	8 140	140
4	22 120	28 ×	70 ×	28 (80)	200
No. bags of retardant required	120	50	50	280	Σ500

Figure 5.—*After the fourth assignment is made, only one column (crew No. 4) is still to be satisfied. The remaining allocations of retardant can be made by inspection. All bags left in warehouses 1, 2, and 4 are assigned to crew 4.*

Two additional rules may be helpful:

1. *When the greatest difference is obtained at the same time in both a row and a column, and the square at the junction of that row and column has the lowest cost, then the assignment is made at that square common to both row and column. If the square at the junction does not have the lowest cost, then make an assignment in either row or column where the lowest cost exists.*
2. *When the largest difference is obtained at the same time in two or more rows (or columns), then assign wherever the smallest costs exist. When a row and a column are simultaneously fulfilled with one assignment, the problem is considered* degenerate. *A degenerate problem may yield several suboptimum solutions unless one of several correction techniques is used. Because VAM provides an approximation only, degeneracy is not a serious problem with its use.*

Our elementary problem has shown an efficient, although not necessarily optimum, assignment of resources. Extensions of the VAM technique can lead not only to more accurate solutions but to more refinement. For example, what types of truck should be used to haul the various loads? Note that the 10 bags going to crew 4 would not even be a full pickup load.

ASSIGNMENT MODEL

The second of the two mathematical techniques described in this note is called the *assignment model.* It is useful if you want to associate each number of origins with each of the *same* number of destinations. Each origin is to be associated with only one destination, and in such a way as to minimize the total cost (or, in some cases, to maximize total effectiveness). You may, for example, want to assign 10 men to 10 jobs so their total skills most effectively match that required by the jobs. The associations can be shown in a *square matrix*—the number of rows equals the number of columns (fig. 6).

Supply Equals Demand

Eight crews of four or five each are to be housed at fire crew stations. Each crew will be assigned to prepare a series of clearcut blocks or projects and burn them. Crews are to be assigned to projects in such a way that minimizes travel time and distance. The assignment of crews to projects and the round-trip distances involved have 40,320 possibilities. Trying to solve this problem in your mind would be out of the question.

The technique used to solve this problem was developed by Flood.[3,4,5] It consists of 11 steps:

1. *Examine the columns of the matrix and determine the smallest element in each column (fig. 6).*
2. *Form a new matrix by subtracting this smallest element from each of the values in the columns. The new matrix will now have one or more zeros in each column.*
3. *Find the minimum number of lines that can be drawn through all columns or rows with zeros (fig. 7). If the number of lines equals the number of rows or columns, we can skip to step 11. If it does not, proceed to step 4.*

The minimum number of lines that could cover all zeros (nul elements) in the matrix is 7 (fig. 7). The number of rows or columns is 8; therefore we must go to step 4.

Project / Crew No	1	2	3	4	5	6	7	8	
1	79	15	36	17	26	38	77	51	
2	44	28	16	64	40	43	87	91	
3	23	77	23	18	92	56	99	15	
4	76	23	72	12	86	82	38	62	
5	76	79	53	16	36	36	3	22	
6	74	39	88	37	13	77	67	93	
7	39	30	63	53	32	8	8	42	
8	37	35	69	69	1	20	98	18	
Least	23	15	16	12	1	8	3	15	

Figure 6.—*Assignment model No. 1. Matrix shows round trip distances between crew headquarters and slash projects. Find the smallest value in each column and enter it in the bottom row.*

Project / Crew No	1	2	3	4	5	6	7	8	Least
1	56	0	20	5	25	30	74	36	0
2	21	13	0	52	39	35	84	76	0
3	0	62	7	6	91	48	96	0	0
4	53	8	56	0	85	74	35	47	0
5	53	64	37	4	35	28	0	7	0
6	51	24	72	25	12	69	64	78	12
7	16	15	47	41	31	0	5	27	0
8	14	20	53	57	0	12	95	3	0

Figure 7.—*Steps 2, 3, and 4. Seven lines will cover all of the zeros, but the matrix has 8 columns or rows.*

Project / Crew No	1	2	3	4	5	6	7	8	
1	56	0	20	5	25	30	74	36	
2	21	13	0	52	39	35	84	76	
3	0	62	7	6	91	48	96	0	
4	53	8	56	0	85	74	35	47	
5	53	64	37	4	35	28	0	7	
6	39	12	60	13	0	57	52	66	
7	16	15	47	41	31	0	5	27	
8	14	20	53	57	0	12	95	(3)	

Figure 8.—*Steps 5, 6, and 7. The smallest value not covered by any line is 3, circled under crew 8, Project 8. The aim is to cover the zeros with the* fewest *lines.*

4. *Examine the rows of the matrix, identifying the smallest element in each row.*
5. *Form a new matrix by subtracting this element from each of the values in the row. The new matrix will now have at least one zero in each row (fig. 8).*
6. *Follow step 3, denoting the minimum set of lines that can be drawn through the zeros.*

There are still seven lines–one short of the number of lines needed to skip to step 11. (Note: A single vertical line covers the two zeros in column 5, fig. 8, whereas two horizontal lines would have been required to do the job.)

7. *Examine closely the matrix and find the smallest value not covered by any line; call it* h.

The smallest value not covered by a line is 3 miles–the distance between crew 8 and project 8.

8. *Form a new matrix (fig. 9) by applying this value the following way:*
   a. *Subtract* h *from all elements not covered by any line.*
   b. *Add* h *to all elements covered by two lines, i.e., at the intersection of a vertical and horizontal line.*
   c. *Make no change in values covered by a single line.*
9. *Following the procedure in step 3, denote the minimum set of lines that can be drawn through the zeros. Go to step 10 or 11, depending upon the number of lines.*
10. *Repeat steps 7 through 9 until the number of lines equals the number of rows or columns.*
11. *Determine the independent zeros by examining all zeros and selecting a combination in which no two will be in the same row or column (fig. 10). The location of independent zeros, denoted by brackets, show the optimum solution.*

Crew No \ Project No	1	2	3	4	5	6	7	8
1	56	0	20	5	28	30	74	36
2	21	13	0	52	42	35	84	76
3	0	62	7	6	94	48	96	0
4	53	8	56	0	88	74	35	47
5	53	64	37	4	38	28	0	7
6	36	9	57	10	0	54	49	63
7	16	15	47	41	34	0	5	27
8	11	17	50	54	0	9	92	0

Figure 9.–*Steps 8 and 9. After subtracting h = 3 from the matrix in Figure 8, the number of lines required to cover all zeros = 8 – the number of rows and columns.*

Crew No \ Project No	1	2	3	4	5	6	7	8
1	56	[0]	20	5	28	30	74	36
2	21	13	[0]	52	42	35	84	76
3	[0]	62	7	6	94	48	96	0
4	53	8	56	[0]	88	74	35	47
5	53	64	37	4	38	28	[0]	7
6	36	9	57	10	[0]	54	49	63
7	16	15	47	41	34	[0]	5	27
8	11	17	50	54	0	9	92	[0]

Figure 10.–*Location of independent zeros (in brackets) indicates the location of the optimum assignment. There is only one zero in each row and column. Some zeros, for example row 8 column 5, are not independent since they occur in rows or columns with other zeros.*

Flood's technique is an optimization of the problem. The payoff in using his method would have been handsome indeed. For example, in several tries, not using the step-by-step procedure, our best solution was 119 miles per day. The optimum solution, using allocation technique is 108 miles–a daily saving of 11 miles. This does not sound too impressive until we estimate that we will save daily:

11 miles at \$0.50 per mile	= \$ 5.50*
30 min. x 5 men at \$2.30/hr.	= 5.79+
Total	\$11.20

* This considers an operating fleet of crew trucks, fire trucks, and an occasional special purpose vehicle.

\+ First step GS-4 salary scale.

If the project runs for 16 weeks, the savings can amount to \$896–a good return on a few minutes invested in using allocation technique.

Supply Less Than Demand

In the first two examples, supply equaled demand. But we know, from real life, that this is not always the case. In the following example, the assignment model is modified to solve a problem in which supply is less than demand. Destinations must share supplies in such a way that minimizes conflicts. Four fire trucks are to be assigned to eight crews in the most efficient (but necessarily optimum) way. The crews are to share trucks in such a way that minimizes overall conflicting needs.

Developed by Joseph, the model is called the Heuristic Approach to Assignment Problems (HAAP).[6]

Suppose that a crew knows that it will need a fire truck sometime during the week but not which day. Then the odds are 1/5, or 0.2, of needing the truck any one day. There is the probability of 1/5 x 5, or 1, of needing it during the week (we are basing this on 5-day weeks without holidays–if a holiday occurred, we would base the probability on 4, rather than 5 days).

Suppose that a second crew also needed the truck for an unspecified day during the same week. The odds would then be 1/5 x 1/5 = 1/25 = 0.04 that both crews would need it on a given day or 1/25 x 5 = 1/5, or 0.2, of a conflict occurring during the week. The odds are pretty good that there will be no problem. Now, suppose, on the other hand, that the second crew needed the truck 4 days a week. The odds would be 1/5 x 4/5 = 4/25 = .16 of conflict on any specific day; however, the chances are 4/25 x 5 = 20/25 = .8 of conflict during the week. Obviously, there will be less conflict if a truck is shared under condition 1 rather than 2. While this is intuitive in a small example, it can become exceedingly complex as the problem increases in resources and demands, and extends over a long time period.

Now let us assume that for each day a crew is as likely to require a fire truck as any other day. Then $h_{ir}/5$ is the probability that the *i*th crew will require the truck on the *r*th day. If another crew, the *j*th, is also considered, $h_{jr}/5$ is the probability that the *j*th crew will need the truck on the *r*th day. Then $h_{ir}/5$ x $h_{jr}/5$ is the probability that both the *i*th and *j*th crew will demand the truck on any given day. This last number times 5, or h_{ir} x $h_{jr}/5$ is the expected number of days of conflict during the week in question.

If the number of days of simultaneous demand or conflict by the *i*th and *j*th crews is summed and divided by the number of working days, the result is the average number of days of weekly conflict between the *i*th and *j*th crew during the 16-week period (fig. 11).

Weeks / Crew No.	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
1	1	5		2		1		5	5	5	2	1				5
2	5	1		4	2	5	5	2	1	1	1	1	1	4	5	2
3	5	5	5	5			1	1	1	1	2	3	4	5	2	1
4		2	1	1				4	5	2	1	1	4	5	2	1
5								2	3	2	1	1	4	5	2	1
6						5	5	5	5							
7	1	2	3	2	1			4	5	5	5	1	1	1	1	1
8	5					4	5	2	1	1	1	1	2	3	2	3

Figure 11.–*The weekly needs of each crew for a fire truck. The figure can be prepared from crew foreman estimates of individual crew needs.*

The five steps in the HAAP are:

1. *Multiply the* r*th period requirement for the* i*th unit with that of the* j*th unit for each of the* r *periods and sum.*

$$(P_{ij} = \sum_{r=1}^{r=16} h_{ir}h_{jr})$$

Enter this value in the appropriate ij *cell of the conflict matrix (fig. 12). Notice that the matrix is symmetric and that the* h_{ij} *when* i = j *is taken to be zero.*

Crew No. / Crew No.	1	2	3	4	5	6	7	8	Ki	Re-order
1	0	56	67	75	43	55	101	47	451	3
2	56	0	100	68	47	65	48	108	487	1
3	67	100	0	68	48	15	79	69	446	4
4	75	68	68	0	75	41	78	47	452	2
5	43	47	48	75	0	25	51	41	340	7
6	55	65	15	41	25	0	45	60	286	8
7	101	48	79	78	51	45	0	39	441	5
8	47	108	69	47	41	60	39	0	411	6

Figure 12.–*Assignment model No. 2. Steps 1 and 2 shows the level of "conflict" among crews for available fire trucks.* K_i *is the sum of the conflicts for a particular crew.*

In our example, the weekly and total conflict for crew 1 and 2 are computed in the following manner:

Week	*Crew 1*	*x*	*Crew 2*	=	*Product*	
1	1		5		5	
2	5		1		5	
3	0		0		0	
4	2		4		8	
5	0		2		0	
6	1		5		5	
7	0		5		0	†
8	5		2		10	
9	5		1		5	
10	5		1		5	
11	2		1		2	
12	1		1		1	
13	0		1		0	
14	0		4		0	
15	0		5		0	
16	5		2		10	
					56††	

† ÷ 5 = Number of days during a given week in which a conflict can be expected.

†† ÷ 80 = Average number of days of conflict expected per week during 16-week period.

The value–56–is entered under crew 1 and 2. Average conflict values are found for all of the crew relationships.

2. *Sum the individual conflicts for each unit with every other unit along the row.*

$$(K_i = \sum_{j=1}^{j=8} P_{ij})$$

Crew 1's conflict equals 56 + 67 + 75 + 43 + 55 + 101 + 47 = 451. This is entered in the second column from the right.

3. *Construct a new matrix ordering the units so that the largest conflict sum* (K_i) *is first and the smallest value last (fig. 13). Resources–shown as column headings–are assigned in any sequence as long as they are available without conflict.*

The crews are ordered so that the crew (No. 2) that conflicts most is first and the crew (No. 6) that conflicts the least is in last place. The four fire trucks are distributed to the four top crews–one to each. Crews 4, 2, 1, and 3 are taken care of but we still need to set up a sharing arrangement for crews 7, 8, 5, and 6.

4. *Make assignment to the highest initially unassigned unit by determining the lowest conflict (from the conflict matrix–fig. 12–with previously assigned units).*

The highest initially unassigned crew–crew 7 (K_i = 441)–conflicts with the initially assigned crews:

1 – 101 3 – 79

2 – 48 4 – 78

The lowest daily conflict occurs with crew 2; therefore, crew 7 and crew 2 will share a truck.

5. *Repeat step 4 (until all units are assigned) considering that the highest unassigned crew may now share a resource with one or more assigned units.*

Crew No.	Ki	Truck No. 1	2	3	4	Cost
2	487	×				0
4	452		×			0
1	451			×		0
3	446				×	0
7	441		×			(48), 78, 101, 79
8	411				×	(47), (108+39), 69, 47
5	340	×				(43), (47+51), 48, (75+41)
6	286			×		(55+25), (65+45), (15), (41+60)

Figure 13.–*Assignment matrix. The X's mark the sharing arrangement for each truck, i.e., truck 1 is shared by crews 2 and 5. The odds are slightly more than half (48/80) that there will be a conflict during an "average" week.*

The next highest initially unassigned crew–crew 8 (K_i = 411)–conflicts with the previously assigned crews (crew 7 is now included). Conflict costs are:

1 – 47

2 & 7 – (108 + 39)

3 – 69

4 – 47

Least conflict is with crew 1 and 4 = 47. Crew 2 and 7 are already sharing a truck, and the conflict would be the *sum* of conflicts with 2 and 7. The procedure is continued until all crews are assigned.

There are additional ways of reducing such a conflict if the opportunity exists. For example, with items as mobile as trucks, it might be more practical to use a pool rather than an assignment method. In fact, such an analysis could lead to a decision on whether to pool or to assign vehicles. A second way might be to assign them for a shorter period of time–say, 3 or 4 weeks. A third method, if conditions permitted, would be the scheduling or sequencing of the burns or other project activities.

Allocation techniques will pay off in proportion to the ingenuity of their use. I have discussed their use in terms of men, fire trucks, and slash burning. I could just as well have used any of the countless things that foresters need to assign or put some place.

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